JOURNAL OF MULTIVARIATE ANALYSIS 28, 88–114 (1989)

Rank Tests for Matched Pair Experiments with Censored Data*

DOROTA M. DABROWSKA[†]

Carnegie-Mellon University and University of California, Berkeley

Communicated by the Editors

We consider the problem of testing bivariate symmetry in matched pair experiments where (X_1, X_2) are time measurements such as failure or survival times. The observations are subject to random right censoring so that what is observed is $Y_j = \min(X_j, Z_j)$ and $\delta_j = I(X_j = Y_j)$, $j = 1, 2$, where (Z_1, Z_2) is a pair of censoring times independent of (X_1, X_2). Tests that generalize the conditional Wilcoxon and the log-rank tests are considered as well as general linear rank statistics. It is shown that suitably standardized versions of these statistics are asymptotically normal under fixed and converging alternatives and they are consistent against the alternative of ordered hazards. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let $X_i = (X_{1i}, X_{2i})$ and $Z_i = (Z_{1i}, Z_{2i})$, $i = 1, ..., n$ be mutually independent sets of nonnegative bivariate random variables (rv) defined on a common probability space. The X_i's and Z_i's are independent identically distributed (iid) rv's with continuous joint distribution functions (cdf) F and G, respectively, and marginal cdf's F_1, F_2 and G_1, G_2. For each $i = 1, ..., n$, the observable rv's are given by $Y_i = (Y_{1i}, Y_{2i})$ and $\delta_i = (\delta_{1i}, \delta_{i2})$, where $Y_{ji} = \min(X_{ji}, Z_{ji})$, $\delta_{ji} = I\ (X_{ji} = Y_{ji})$, and $I(A)$ is the indicator function of the set A. The variables X_{1i} and X_{2i} are thought of survival or failure times. For each subject we observe his survival time X_{ji} or censoring time Z_{ji}, $j = 1, 2$, whichever occurs first, together with a random variable δ_{ji} indicating if he has left the study due to death or withdrawal. Examples of this kind of censoring mechanism have been considered by several authors. Clayton [8], for instance, discusses a model to study the familial tendency

Received March 22, 1987; revised November 5, 1987.
AMS 1980 subject classifications: 62G10, 62H15.
Key words and phrases: censored data, tests for symmetry, rank tests.
* This research was supported by the University of California Presidential Fellowship and the National Institute of General Medical Sciences Grant SSS-Y1RO1 GM35416-02.
[†] Current address: University of California, Los Angeles.

0047-259X/89 $3.00

in chronic disease incidence. For each father–son pair, X_1 and X_2 denote the father's and his son's age at the onset of the disease. Then X_1 and X_2 are observable unless the father or his son withdraws from the study. Further examples of this type of censoring can be found in Langberg and Shaked [18], Tsai *et al.* [28], Campbell [5, 6], Clayton and Cuzick [9], Oakes [21], and Wei and Pee [30].

The paper deals with the problem of testing the hypothesis of bivariate symmetry of the survival times H_0: (X_1, X_2) has the same distribution as (X_2, X_1), against the alternative hypothesis that the distribution of (X_1, X_2) is asymmetric in such a way that X_1 tends to assume larger values than X_2. This testing problem was discussed extensively by Schaafsma [24], Snijders [26, 27], Bell and Haller [2], Yanagimoto and Sibuya [31], and Doksum [13], among others.

Here we consider tests based on ranks of X_{1i} and X_{2i} in the pooled sample $X_{11}, X_{21}, ..., X_{1m}, X_{2n}$. These ranks arise from invariance considerations when we tests the hypothesis H_0 against the alternative H_1: $P(X_{1i} \leqslant h(X_{2i})) \geqslant P(X_2 \leqslant h(X_{1i}))$ for all continuous increasing functions h. In Section 2 we discuss a Hoeffding type formula for the distribution of the censored data rank vector under arbitrary bivariate distribution. This leads to construction of locally most powerful conditional rank tests. The resulting tests are based on the same statistics as in the case of univariate two-sample problem (see, e.g., Prentice [22] and Kalbfleisch and Prentice [16]). However, the critical values are obtained by conditioning on the particular configuration of ranks. This leads to conditional similar rank tests.

In the presence of censoring, the practical evaluation of exact critical values of these conditional tests does not seem to be feasible especially when the censoring is heavy. In Section 3 we discuss asymptotic distribution of the corresponding unconditional tests and show that these unconditional tests are consistent against the alternative of ordered hazards.

Our approach to the asymptotic distribution theory patterns the Chernoff–Savage [7] and Pyke–Shorack [23] approach to the asymptotic distribution of two-sample rank statistics for uncensored data. Suitably standardized versions of the test statistics are shown to be asymptotically normal under arbitrary fixed and converging alternatives. The results are used to derive efficacies of the tests under contiguous alternatives. An estimator of the asymptotic null variance is provided.

2. Conditional Censored Data Rank Tests

We start with uncensored data and follw the ideas of Snijders [26, 27] and Doksum [13]. Let $R_{11}^*, ..., R_{1n}^*$ and $R_{21}^*, ..., R_{2n}^*$ denote the ranks of

$X_{11}, ..., X_{1n}$ and $X_{21}, ..., X_{2n}$ among $X_{11}, X_{21}, ..., X_{1n}, X_{2n}$. Further, for each $i = 1, ..., n$ set $R^*_{(1)i} = \max(R^*_{1i}, R^*_{2i})$, $R^*_{(2)i} = \min(R^*_{1i}, R^*_{2i})$. Suppose that the joint distribution of (X_1, X_2) has density $f_\theta(s, t)$, where $\theta > 0$ and let the hypothesis of bivariate symmetry correspond to $\theta = 1$. For $\theta = 1$, we have $f_1(s, t) = f_1(t, s)$ and let h be the common marginal density of X_1 and X_2. Further, let H be the corresponding distribution function. From Snijders [26, 27] we find the following Hoeffding type formula for the conditional probability of R^* given $R^*_{(\,)}$.

LEMMA 2.1. *If the family $\{f_\theta(s, t): \theta > 0\}$ is dominated by $h(s)\,h(t)$ then*

$$P_\theta(R^* = r \mid R^*_{(\,)} = r_{(\,)}) = \frac{E \prod_{i=1}^n \Phi(H^{-1}(U_{(r_{1i})}), H^{-1}(U_{(r_{2i})}); \theta)}{2^k E \prod_{i=1}^n \bar{\Phi}(H^{-1}(U_{(r_{1i})}), H^{-1}(U_{(r_{2i})}); \theta)}.$$

Here $U_{(1)} < \cdots < U_{(2n)}$ is an ordered sample of size $2n$ from the uniform distribution on $(0, 1)$, $\Phi(s, t; \theta) = h(s)^{-1} h(t)^{-1} f_\theta(s, t)$, $\bar{\Phi}(s, t; \theta) = \{\Phi(s, t; \theta) + \Phi(t, s; \theta)\}/2$ and $k = \#\{i: r_{1i} \neq r_{2i}\}$.

Tests for bivariate symmetry can be now based on score statistics corresponding to $P_\theta(R^* = r)$. In particular, if (X_1, X_2) are independent under the null hypothesis ($\theta = 1$) the resulting tests reject the hypothesis for large values of

$$\sum_{i=1}^n [a(R^*_{1i}) - a(R^*_{2i})],$$

where a is an appropriate score function. The resulting tests look like tests for the usual two sample problem with equal simple size, the difference is that the critical values are determined now from the distribution of $R^*_{(\,)}$. The tests are conditionally distribution free in the sense that given the values of $R^*_{(\,)}$, under the hypothesis $r_{(1)i}$ is equally likely to be the rank of X_{1i} or the rank of X_{2i} (see Snijders [26, 27] and Doksum [13]).

In the presence of censoring, we define censored data ranks as in Prentice [22] and Kalbfleisch and Prentice [16]. More precisely, let

$$R_{1i} = \sum_{j=1}^n [\delta_{1j} I(Y_{1j} \leqslant Y_{1i}) + \delta_{2j} I(Y_{2j} \leqslant Y_{1i})]$$

$$R_{2i} = \sum_{j=1}^n [\delta_{1j} I(Y_{1j} \leqslant Y_{2i}) + \delta_{2j} I(Y_{2j} \leqslant Y_{2i})].$$

Thus uncensored observations are ranked among themselves and each censored observation is assigned the same rank as the nearest uncensored observation on the left. For $j = 1, 2$, let $n_j = \sum_{i=1}^n \delta_{ji}$ be the observed number of uncensored observations among Y_{ji}'s, $i = 1, ..., n$. Further, for each

$d=(d_1, d_2)$ with $d_j=0$ or 1, let $A_d=\{i: \delta_{1i}=d_1, \delta_{2i}=d_2\}$. The values of n_j and A_d characterize the observed pattern of deaths and withdrawals. The censored data rank set is now thought of as the collection $\mathbf{R}$ of all possible rankings (R^*_{1i}, R^*_{2i}) of (X_{1i}, X_{2i}) in the uncensored version of the experiment that is compatible with the observed values of n_j, A_d, and (r_{1i}, r_{2i}), $i=1, ..., n$. Let $\mathbf{R}_{(\,)}$ be the set of all possible ordered rankings $(R^*_{(1)i}, R^*_{(2)i})$ of (X_{1i}, X_{2i}), where $\{(R^*_{1i}, R^*_{2i})\colon i=1, ..., n\} \in \mathbf{R}$. The conditional distribution of $\mathbf{R}$ given $\mathbf{R}_{(\,)}$ depends in a complicated way on the distribution of both survival and censoring variables. Following Prentice [22] and Kalbfleisch and Prentice [16], we give a Hoeffding type formula for the conditional distribution of $\mathbf{R}$ given $\mathbf{R}_{(\,)}$ appropriate for the uncensored version of the experiment, i.e., given the observed pattern of deaths and withdrawals.

LEMMA 2.2. *Let the assumption of Lemma 2.1 be satisfied. In the uncensored version of the experiment, the conditional distribution of* $\mathbf{R}$ *given* $\mathbf{R}_{(\,)}$ *and given the observed pattern of deaths and withdrawals is*

$$P_\theta(\mathbf{R} \mid \mathbf{R}_{(\,)}, A)=\frac{E\prod_i \Phi_{\delta_i}(H^{-1}(U_{(r_{1i})}), H^{-1}(U_{(r_{2i})}); \theta)}{2^k E\prod_i \bar{\Phi}_{\delta_i}(H^{-1}(U_{(r_{1i})}), H^{-1}(U_{(r_{2i})}); \theta)}.$$

Here $A=\{A_{00}, A_{10}, A_{01}, A_{11}\}$, $U_{(1)}<\cdots<U_{(n_1+n_2)}$ *an ordered sample of size* n_1+n_2 *from uniform distribution on* $(0, 1)$ *and* $k=\#\{i: r_{1i}\neq r_{2i}\}$. *Furthermore,* $\bar{\Phi}_d(s, t; \theta)=\{\Phi_d(s, t; \theta)+\Phi_d(t, s; \theta)\}/2$ *and*

$$\begin{aligned}
\Phi_d(s, t; \theta) &= f_1(s, t)^{-1} f_\theta(s, t) && \text{if } d=(1, 1)\\
&= f_{11}(s)^{-1}\int_t^\infty f_\theta(s, u)\, du && \text{if } d=(1, 0)\\
&= f_{12}(t)^{-1}\int_s^\infty f_\theta(u, t)\, du && \text{if } d=(0, 1)\\
&= \int_s^\infty \int_t^\infty f_\theta(u, v)\, du\, dv && \text{if } d=(0, 0).
\end{aligned}$$

Here f_{11} *and* f_{12} *denote the marginal densities of* X_1 *and* X_2, *respectively, corresponding to the density* $f_1(s, t)$.

The lemma follows from Lemma 2.1 and arguments similar to Kalbfleisch and Prentice [16, p. 154]. We omit the details.

Tests for bivariate symmetry can be derived as score statistics corresponding to (2.2). Following Doksum [13], we consider the generalized scale model as a special example. Here

$$X_{1i}=\eta_i+(\theta-1)\,\varepsilon_i \qquad X_{2i}=\theta\tilde{\eta}_i+(\theta-1)\,\varepsilon_i, \tag{2.1}$$

where η_i and $\tilde{\eta}_i$, $i=1, ..., n$ are mutually independent samples from distribution function H, $H(0)=0$, and ε_i, $i=1, ..., n$ is a sample from the distribution function M, independent of η_i's and $\tilde{\eta}_i$'s. A straightforward calculation shows that the joint density of (X_{1i}, X_{2i}) is given by

$$f_\theta(s, t)=\theta^{-1} \int h(s-(\theta-1)e)\, h(\theta^{-1}[t-(\theta-1)e])\, dM(e),$$

where h is the density of H. For $\theta=1$, $f_1(s, t)=h(s)\, h(t)$. Under suitable regularity conditions (Hájek and Šidák [15, p. 70]), the scores test corresponding to (2.1) rejects the hypothesis for large values of

$$\sum_{i=1}^{n} [a(R_{1i}, \delta_{1i})-a(R_{2i}, \delta_{2i})],$$

where

$$a(i, d)=2^{-1}EJ(U_{(i)}, d) \prod_{k=1}^{n_1+n_2} m_k(1-U_{(k)})^{\alpha_k}. \tag{2.2}$$

Here $\alpha_k=\#\{i: R_{ji}=k,\ \delta_{ji}=0,\ j=1, 2\}$, $m_k=\#\{i: R_{ji}\geqslant k,\ j=1, 2\}$, and

$$\begin{aligned} J(u, d) &= -[1+H^{-1}(u)\, h'(H^{-1}(u))/h(H^{-1}(u))] && \text{if} \quad d=1 \\ &= H^{-1}(u)\, h(H^{-1}(u))/(1-u) && \text{if} \quad d=0. \end{aligned}$$

This type of score was extensively studied in the survival analysis literature in the context of the usual two-sample problem. See, for instance, Prentice [22], Kalbfleisch and Prentice [16]. It can be easily verified that the score generating function J satisfies

$$\int_0^u J(v, 1)\, dv = -(1-u)\, J(u, 0). \tag{2.3}$$

The choice of standard exponential H, leads to $J(u, d) = -d-\ln(1-u)$. The resulting test is the log-rank test based on the statistic

$$T_n=(2n)^{-1} \sum_{i=1}^{n} [\hat{\Lambda}(Y_{2i})-\delta_{2i}-\hat{\Lambda}(Y_{1i})+\delta_{1i}],$$

where $\hat{\Lambda}$ is the Aalen–Nelson estimator (Aalen [1], Nelson [20])

$$\hat{\Lambda}(t)=\sum_{s\leqslant t} \frac{\Delta\hat{K}(s)}{1-\hat{H}(s-)}, \tag{2.4}$$

where $\Delta\hat{K} = (\Delta\hat{K}_1 + \Delta\hat{K}_2)/2$, $\hat{H} = (\hat{H}_1 + \hat{H}_2)/2$, $\Delta\hat{K}_j(s) = n^{-1}\sum_i I(Y_{ji} = s, \delta_{ji} = 1)$, $\hat{H}_j(s) = n^{-1}\sum_i I(Y_{ji} \leqslant s)$. The choice of loglogistic H, leads to $J(u, d) = (1 + d)\,u - d$. The resulting test is the censored data analog of the conditional Wilcoxon rank test based on the statistic

$$U_n = (2n)^{-1} \sum_{i=1}^{n} [(1 + \delta_{2i})\,\hat{S}(Y_{2i}) - \delta_{2i} - (1 + \delta_{1i})\,\hat{S}(Y_{1i}) + \delta_{1i}],$$

where $\hat{S}$ is an estimator close to the Kaplan–Meier estimator [17]

$$\hat{S}(t) = 1 - \prod_{s \leqslant t} \left(1 - \frac{\Delta\hat{K}(s)}{1 - \hat{H}(s-) + (2n)^{-1}}\right). \tag{2.5}$$

In general, the exact scores (2.2) might be hard to compute. Therefore, following Prentice [22], Kalbfleisch and Prentice [16], Cuzick [11], and Dabrowska [12], we shall consider approximate score statistics

$$U_n = (2n)^{-1} \sum_i [J(\hat{S}(Y_{2i}), \delta_{2i}) - J(\hat{S}(Y_{1i}), \delta_{1i})],$$

where $\hat{S}$ is given by (2.5) and the score functions J satisfy the integral equation (2.3).

The practical evaluation of exact critical values of these conditional tests does not seem to be feasible. In the following section, we discuss the asymptotic distribution of the unconditional tests and show that these unconditional tests are consistent against the alternative of ordered hazards.

3. Asymptotic Distributions: Assumptions and Results

First let us introduce some assumptions to be used throughout this and subsequent sections.

A.1. *For each $n = 1, 2, ..., (X_{1i}, X_{2i})$ and (Z_{1i}, Z_{2i}), $i = 1, ..., n$ are mutually independent sets of iid nonnegative bivariate rv's with continuous joint cdf's F_n and $G_n = G$ and marginal cdf's F_{n1}, F_{n2} and G_1, G_2. For some continuous cdf F, $F_n \to F$ as $n \to \infty$.*

For each n define $L_n(s, t, d_1, d_2) = P(Y_{1i} \leqslant s,\ Y_{2i} \leqslant t,\ \delta_{1i} \leqslant d_1,\ \delta_{2i} \leqslant d_2)$ and for $j = 1, 2$ let $L_{nj}(s, d) = P(Y_{ji} \leqslant s,\ \delta_{ji} \leqslant d)$, $H_{nj}(s) = P(Y_{jn} \leqslant s)$, and $K_{nj}(s) = 1 - P(Y_{ji} > s,\ \delta_{ji} = 1)$. Under assumption A.1, these cdf's may be easily expressed in terms of F_n and G. Moreover, L, L_j, H_j, and K_j, their limiting distributions, exist and depend on F and G only. Finally, let $\hat{L}$, $\hat{L}_j$, $\hat{H}_j$, and $\hat{K}_j$ denote the corresponding empiricals.

The proof of the asymptotic normality of suitably standardized versions of T_n and U_n rests on a decomposition into sums of leading terms which are asymptotically normal, and remainder terms, which are asymptotically negligible. As regards the statistic U_n, we assume that the score generating function J satisfies the following smoothness and boundedness conditions.

A.2. *For $d=0, 1$, $J(u, d)$ is a continuously differentiable function on $[0, 1)$ such that $|J(u, d)| \leqslant cr(u)^a$ and $|J'(u, d)| \leqslant cr(u)^b$ where $r(u) = (1-u)^{-1}$ and $c>0$, $0<a$, $b<\frac{1}{2}$.*

Define

$$\Lambda_n(t) = \int_0^t \frac{dK_n}{1-H_n}$$

and

$$S_n(t) = 1 - \exp\{-\Lambda_n(t)\},$$

where $K_n = (K_{n1} + K_{n2})/2$, $H_n = (H_{n1} + H_{n2})/2$. Furthermore, let $\Lambda(t) = \lim \Lambda_n(t)$ and $S(t) = \lim S_n(t)$. Set

$$A_{1n} = n^{1/2} 2^{-1} \int J(S_n(y), d)\, d(\hat{L}_2 - L_{n2})(y, d)$$

$$A_{2n} = -n^{1/2} 2^{-1} \int J(S_n(y), d)\, d(\hat{L}_1 - L_{n1})(y, d)$$

$$A_{3n} = n^{1/2} 2^{-1} \int W_n(y)(1 - S_n(y))\, J'(S_n(y), d)\, dL_{n2}(y, d)$$

$$A_{4n} = -n^{1/2} 2^{-1} \int W_n(y)(1 - S_n(y))\, J'(S_n(y), d)\, dL_{n1}(y, d).$$

Here

$$W_n(y) = \int_0^y (\hat{H}^- - H_n)\, r(H_n)^2\, dK_n + \int_0^y r(H_n)\, d(\hat{K} - K_n).$$

LEMMA 3.1. *Let the assumption* A.1 *be satisfied and let J be a function such that* A.2 *holds with $0 < b \leqslant 1$. Then with probably* 1, *$n^{1/2} \sum_{k=1}^4 A_{kn}$ is a sum of iid rv's with mean zero and absolute moment of order $2+\eta$, uniformly bounded above for some $\eta > 0$.*

The proof is deferred to Section 5. To standardize T_n and U_n for location and scale, define

$$\mu_n = \mu(F_n, G) = 2^{-1}E[J(S_n(Y_2), \delta_2) - J(S_n(Y_1), \delta_1)]$$

$$\sigma_n^2 = \sigma^2(F_n, G) = \operatorname{var}\left(\sum_{k=1}^{4} A_{kn}\right).$$

Under conditions of Lemma 3.1, σ_n^2 is well defined and converges to $\sigma_0^2 = \sigma^2(F, G) = \operatorname{var}(\sum_{k=1}^{4} A_{k0})$, where the variance σ_0^2 is evaluated under F and G and the terms A_{k0} are defined as A_{kn} with S_n, H_n, K_n, and L_{nj} replaced by their limiting distributions. Further, with probability 1,

$$n^{1/2}(T_n - \mu_n) = \sum_{k=1}^{4} A_{kn} + B_n \tag{3.1}$$

$$n^{1/2}(U_n - \mu_n) = \sum_{k=1}^{4} A_{kn} + C_n, \tag{3.2}$$

where B_n and C_n are remainder terms.

THEOREM 3.1. *Let the assumptions* A.1 *and* A.2 *be satisfied. Suppose that* $\sigma_0^2 > 0$ *for* $J(u, d) = -d - \ln(1-u)$ *or* J *satisfying* A.2. *Then* $n^{1/2}(T_n - \mu_n)$ *and respectively* $n^{1/2}(U_n - \mu_n)$ *converge in distribution to* $N(0, \sigma_0^2)$.

The proof of the theorem is given in subsequent sections. In general, the asymptotic variance of T_n and U_n depends in a complicated way on the underlying joint distributions of both survival and censoring times. We consider now the case of the null hypothesis H_0: $F(s, t) = F(t, s)$ in more detail.

Under the null hypothesis, if the integral equation (2.3) is satisfied then Lemma 4.1 and assumption A.1 entail $S = F_1 = F_2$ and $E[J(S(Y_j), \delta_j) \mid Z_j] = 0$, so that the asymptotic null mean is equal to zero. Furthermore, a simple calculation shows that if (2.3) holds then in the case of the statistic U_n the asymptotic null variance is equal to

$$\begin{aligned}\sigma_{0U}^2 &= 4^{-1}E[J(S(Y_{1i}), \delta_{1i}) - J(S(Y_{2i}), \delta_{2i})]^2 \\ &= 4^{-1}\{E\tilde{J}(S(Y_{1i}))^2\, \delta_{1i} + E\tilde{J}(S(Y_{2i}))^2\, \delta_{2i} \\ &\quad - 2E[J(S(Y_{1i}), \delta_{1i})\, J(S(Y_{2i}), \delta_{2i})]\}.\end{aligned}$$

Here $\tilde{J}(u) = J(u, 1) - J(u, 0)$ and $S = F_1 = F_2$. If in addition $F(s, t) = F_1(s)\, F_2(t)$ then the last expectation is equal to 0. In the case of the log-rank statistic the asymptotic null variance is equal to

$$\sigma_{0T}^2 = 4^{-1}\{P(\delta_{1i} = 1) + P(\delta_{2i} = 1) - 2E[(\Lambda(Y_{1i}) - \delta_{1i})(\Lambda(Y_{2i}) - \delta_{2i})]\},$$

where Λ is the cumulative hazard function corresponding to $S = F_1 = F_2$. If in addition $F(s, t) = F_1(s) F_2(t)$, then $\sigma^2_{0T} = 4^{-1}\{P(\delta_{1i} = 1) + P(\delta_{2i} = 1)\}$. In practice, we have to estimate the asymptotic null variance from the data. In the case of the approximate score statistic U_n, set

$$\hat{\sigma}^2_U = (4n)^{-1} \left\{ \sum_{i=1}^{n} \tilde{J}(\hat{S}(Y_{1i}))^2 \delta_{1i} + \sum_{i=1}^{n} \tilde{J}(\hat{S}(Y_{2i}))^2 \delta_{2i} \right.$$
$$\left. - 2 \sum_{i=1}^{n} J(\hat{S}(Y_{1i}), \delta_{1i}) J(\hat{S}(Y_{2i}), \delta_{2i}) \right\}.$$

In the case of the log-rank statistic, set

$$\hat{\sigma}^2_T = (4n)^{-1} \left\{ \sum_{i=1}^{n} \delta_{1i} + \sum_{i=1}^{n} \delta_{2i} - 2 \sum_{i=1}^{n} (\hat{\Lambda}(Y_{1i}) - \delta_{1i})(\hat{\Lambda}(Y_{2i}) - \delta_{2i}) \right\}.$$

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Under the hypothesis of bivariate symmetry,* $\hat{\sigma}^2_U$ *and* $\hat{\sigma}^2_T$ *are consistent estimators of* σ^2_{0U} *and* σ^2_{0T}, *respectively.*

The proof is deferred to Sections 5 and 7.

The following corollary establishes the consistency of the tests against the alternative of ordered hazard functions $H_1 \colon \lambda_1 \geqslant \lambda_2$, where $\lambda_i = f_i/(1 - F_i)$ and f_i is the density of F_i, $i = 1, 2$.

COROLLARY 3.1. *In the case of the statistic* U_n, *assume that the conditions* A.1 *and* (2.3) *are satisfied and let* $\tilde{J}(u) = J(u, 1) - J(u, 0)$ *be a nondecreasing function. The tests* $n^{1/2} U_n/\hat{\sigma}_u$ *and* $n^{1/2} T_n/\hat{\sigma}_T$ *are consistent against* H_1.

The proof is given in Section 5.

Finally, we consider efficacies of these tests. Let $F(s, t)$ be a symmetric cdf and consider the sequence of contiguous alternatives $F_n(s, t)$ given by

$$dF_n(s, t) = \{1 + n^{-1/2} \phi_n(s, t)\} \, dF(s, t),$$

where ϕ_n is a sequence of uniformly bounded functions converging to ϕ, $\phi(s, t) \neq \phi(t, s)$ and

$$\int \phi_n(s, t) \, dF(s, t) = \int \phi(s, t) \, dF(s, t) = 0.$$

Set

$$\phi_{1n}(s) = \int_0^\infty \phi_n(s, t) \, dF_t(s, t) \Big/ \int_0^\infty dF_t(s, t)$$

$$\phi_{2n}(t) = \int_0^\infty \phi_n(s, t) \, dF_s(s, t) \Big/ \int_0^\infty dF_s(s, t),$$

where $d_t F(s, t)$ and $d_s F(s, t)$ stand for integration with respect to t and s, respectively. Then the marginal cdf's F_{n1} and F_{n2} of F_n are of the form

$$dF_{ni}(x) = \{1 + n^{-1/2}\phi_{ni}(x)\}\, dS(x),$$

where $i = 1, 2$ and $S = F_1 = F_2$. Set

$$\Phi_{ni}(x) = \int_x^{\infty} \phi_{ni}(u)\, dS(u).$$

Finally, let ϕ_i and Φ_i be the limits of ϕ_{ni} and Φ_{ni}, $i = 1, 2$.

COROLLARY 3.2. *In the case of the statistic U_n, assume that the conditions* A.1, A.2 *and* (2.3) *are satisfied and let $\tilde{J}(u) = J(u, 1) - J(u, 0)$. The efficacies of the tests based on T_n and U_n are given by*

$$e_T(\phi) = \left\{\int_0^{\infty} \frac{\bar{H}_1 \bar{H}_2}{\bar{H}_1 + \bar{H}_2} [\phi_1 - \phi_2 + (\Phi_1 - \Phi_2)/\bar{S}]\, d\Lambda\right\}^2 \Big/ \sigma_{0T}^2$$

and

$$e_U(\phi) = \left\{\int_0^{\infty} \tilde{J}(S) \frac{\bar{H}_1 \bar{H}_2}{\bar{H}_1 + \bar{H}_2} [\phi_1 - \phi_2 + (\Phi_1 - \Phi_2)/\bar{S}]\, d\Lambda\right\}^2 \Big/ \sigma_{0U}^2,$$

where σ_{0T}^2 and σ_{0U}^2 are the asymptotic null variances of T_n and U_n.

4. PRELIMINARY LEMMAS

In this section, we give a few lemmas which characterize the behaviour of processes $\hat{\Lambda}$ and $\hat{S}$.

LEMMA 4.1. *For $n = 1, 2, ...$ and all t:*

(i) $S_n(t) = \int_0^t (1 - S_n(x-))\, d\Lambda_n(x)$,

$$\hat{S}(t) = \int_0^t (1 - \hat{S}(x-))(1 - \hat{H}(x-) + (2n)^{-1})\, d\hat{K}(x).$$

(ii) $S_n(t) \leqslant H_n(t)$ *and* $\hat{S}(t) \leqslant 2n\hat{H}(t)/(2n+1)$.

(iii) *For all t such that $S_n(t) < 1$,*

$$\hat{S}(t) - S_n(t) = \int_0^t \frac{1 - \hat{S}(x-)}{1 - S_n(x)} \left(\frac{d\hat{K}(x)}{1 - \hat{H}(x-) + (2n)^{-1}} - d\Lambda_n(x)\right).$$

Proof. The proof rests on a repeated application of the following result due to Liptser and Shiryayev [19, p. 255] and Gill [14, p. 153]. If A and B are right continuous nondecreasing functions on R^+, zero at time zero, and $\Delta A \leqslant 1$ and $\Delta B \leqslant 1$, then the unique locally bounded solution Z of

$$Z(t)=\int_0^t \frac{1-Z(x-)}{1-\Delta B(x)}(dA(x)-dB(x))$$

is given by

$$Z(t)=1-\frac{\prod(1-\Delta A(x))\exp(-A_c(x))}{\prod(1-\Delta B(x))\exp(-B_c(x))}, \tag{4.1}$$

where the products are taken over $x \leqslant t$:

(i) The choice of $A(t)=\Lambda_n(t)$, $B(t)\equiv 0$ and an argument similar to the proof of Lemma 3.2.1 in Gill [14] shows the first part of (i). The second follows by setting $A(t)=\int_0^t[1-\hat{H}(x-)+(2n)^{-1}]^{-1}\,d\hat{K}(x)$ and $B(x)\equiv 0$.

(ii) Since $dK_n \leqslant dH_n$, we have by (i) $S_n(t)=1-\exp(-\Lambda_n(t))\leqslant 1-\exp(-\int_0^t dH_n/(1-H_n))=H_n(t)$. Further, a straightforward calculation shows that

$$\frac{2n}{2n+1}\hat{H}(t)=1-\prod_{x\leqslant t}\left(1-\frac{\Delta\hat{H}(x)}{1-\hat{H}(x-)+(2n)^{-1}}\right).$$

Comparing each term of the product with terms appearing in the product defining $\hat{S}(t)$, we obtain $\hat{S}(t)\leqslant 2n\hat{H}(t)/(2n+1)$.

(iii) This follows from (4.1) by setting

$$A(t)=\int_0^t [1-\hat{H}(x-)+(2n)^{-1}]\,d\hat{K}(x) \qquad \text{and} \qquad B(t)=\Lambda_n(x).$$

LEMMA 4.2. *For τ such that $H(\tau)<1$,* $\sup\{|\hat{\Lambda}(t)-\Lambda_n(t)|: 0\leqslant t\leqslant\tau\}\to 0$ *a.s. and* $\sup\{|\hat{S}(t)-S_n(t)|: 0\leqslant t\leqslant\tau\}\to 0$ *a.s.*

The proof is similar to Shorack and Wellner [25, p. 305]. We omit the details.

LEMMA 4.3. *For τ such that $H(\tau)\leqslant 1$, the processes W_n and $(1-S_n)\,W_n$ converge weakly in $D[0,\tau]$ to mean zero Gaussian processes W and $(1-S)\,W$, respectively, and* $\sup_{[0,\tau]}|\hat{\Lambda}-\Lambda_n-W_n|\to_P 0$, $\sup_{[0,\tau]}|\hat{S}-S_n-(1-S_n)\,W_n|\to_P 0$.

The proof of this lemma can be carried out in a fashion similar to Breslow and Crowley [4]. Note however, that since (X_{1n}, X_{2n}) and (Z_{1n}, Z_{2n}) may be pairs of dependent random variables, the covariance structure of W and $(1-S)W$ depends on the joint distributions F and G.

5. Proof of Theorems 3.1 and 3.2: Leading Terms

The proof of Lemma 3.1 rests on a repeated application of inequalities

$$\begin{aligned} &|I(Y_{ji}<x)-H_{nj}(x)|,\ |1-I(Y_{ji}>x,\delta_{ji}=1)-K_{nj}(x)| \\ &\quad \leqslant r(H_{nj}(Y_{ji}))^{1-\gamma}\, r(H_{nj}(x))^{-(1-\gamma)} \end{aligned} \tag{5.1}$$

for $\gamma\in(0,1)$, $j=1,2$ and $i=1,...,n$. Further, Lemma 4.1(i) and A.2 imply

$$\begin{aligned} |J(S_n(x),d)| &\leqslant cr(H_n(x))^a \\ |J'(S_n(x),d)| &\leqslant cr(H_n(x))^b. \end{aligned} \tag{5.2}$$

Proof of Lemma 3.1. We shall show that each of the terms A_{kn}, $k=1,2,3,4$ is a sum iid rv's with mean zero and finite absolute moment of order $2+\eta$ uniformly bounded from above for some $\eta>0$. By symmetry, it is enough to consider the terms A_{1n} and A_{3n}. In what follows, M denotes a generic constant independent of n and the underlying cdf's.

Set $\mu_{1n}=EJ(S_n(Y_{2i}),\delta_{2i})$. We have $n^{1/2}A_{1n}=\sum_{i=1}^n[J(S_n(Y_{2i}),\delta_{2i})-\mu_{1n}]$ which is a sum of iid mean zero rv's. Further, by (5.2)

$$\begin{aligned} E\,|J(S_n(Y_{2i}),\delta_{2i})|^{2+\eta} &\leqslant cEr(H_n(Y_{2i}))^{(2+\eta)a} \\ &= c\int r(H_n(x))^{(2+\eta)a}\,dH_{n2}(x) \\ &\leqslant M\int r(u)^{(2+\eta)a}\,du<\infty \end{aligned}$$

provided $\eta>0$ is chosen so that $a(2+\eta)<1$. This, however, always can be achieved since $a<\frac{1}{2}$. Further, we have $n^{1/2}A_{3n}=\sum_{i=1}^n A_{3i}$, where

$$A_{3i}=\int W_{ni}(x)(1-S_n(x))\,J'(S_n(x),d)\,dL_{n2}(x,d).$$

The process W_{ni} is defined as W_n except that $\hat{H}$ and $\hat{K}$ are replaced by

$\hat{H}_i = (\hat{H}_{1i} + \hat{H}_{2i})/2$ and $\hat{K}_i = (\hat{K}_{1i} + \hat{K}_{2i})/2$, where $\hat{H}_{ji}(x) = I(Y_{ji} < x)$ and $\hat{K}_{ji}(x) = 1 - I(Y_{ji} > x, \delta_{ji} = 1)$, $j = 1, 2$, $i = 1, ..., n$. By (5.2) we have

$$|A_{3i}| \leqslant c \int |W_{ni}(x)|\, r(H_n(x))^a \, dH_{n2}(x).$$

Integration by parts and a little algebra entail that with probability 1, this bound is bounded from above by $\sum_{k=1}^{8} A_{3ik}$, where

$$A_{3i1\perp} = 2^{-1} c \int \left[\int_0^x |\hat{H}_{1i}^- - H_{n1}|\, r(H_n)^2 \, dH_n \right] r(H_n(x))^a \, dH_{n2}(x)$$

$$A_{3i2} = 2^{-1} c \int \left[\int_0^x |\hat{K}_{1i} - K_{n1}|\, r(H_n)^2 \, dH_n \right] r(H_n(x))^a \, dH_{n2}(x)$$

$$A_{3i3} = 2^{-1} c \int |\hat{K}_{1i}(x) - K_{n1}(x)|\, r(H_n(x)) \, dH_{n2}(x)$$

$$A_{3i4} = A_{3i8} = M \int r(H_n)^a \, dH_{n2}.$$

The terms A_{3i5}, A_{3i6}, and A_{3i7} are defined in the same way as A_{3i1}, A_{3i2}, and A_{3i3}, respectively, except that $|\hat{H}_{2i} - H_{n2}|$ and $|\hat{K}_{2i} - K_{n2}|$ replace $|\hat{H}_{1i} - H_{n1}|$ and $|\hat{K}_{1i} - K_{n1}|$. By symmetry, it is enough to consider the terms A_{3i1}, A_{3i2}, and A_{3i3}.

Applying (5.1) with $\gamma = \frac{1}{2} + \eta$, we obtain

$$\begin{aligned} A_{3i1} &\leqslant 2^{-1} c r(H_{n1}(Y_{1i}))^{1/2 - \eta} \int \left[\int_0^x r(H_{n1})^{-1/2+\eta}\, r(H_n)^2 \, dH_n \right] \\ &\quad \times r(H_n(x))^a \, dH_{n2}(x) \\ &\leqslant M r(H_{n1}(Y_{1i}))^{1/2-\eta} \int r(u)^{1-\eta} \, du \int r(u)^{a+1/2+2\eta} \, du. \end{aligned}$$

The $2 + \eta$ moment of the random part on the right-hand side is finite and independent of n because $(\frac{1}{2} - \eta)(2 + \eta) < 1$ for all η. The deterministic part is uniformly bounded from above provided $a + \frac{1}{2} + 2\eta < 1$. The same argument shows that the $2 + \eta$ moment of A_{3i2} is uniformly bounded from above provided $a + \frac{1}{2} + 2\eta < 1$. Further, applying (5.1) with $\gamma = \frac{1}{2} + \eta$,

$$\begin{aligned} A_{3i3} &\leqslant 2^{-1} c r(H_{n1}(Y_{1i}))^{1/2-\eta} \int r(H_n)^{a+1}\, r(H_{n1})^{-1/2+\eta} \, dH_{n2} \\ &\leqslant M r(H_{n1}(Y_{1i}))^{1/2-\eta} \int r(u)^{a+1/2+\eta} \, du \end{aligned}$$

and the same argument as in the case of A_{3i1} shows that the $2+\eta$ moment of A_{3i3} is uniformly bounded from above provided $a+\frac{1}{2}+\eta<1$. Finally,

$$A_{3i4}=A_{3i8}\leqslant M\int r(u)^a\,du<\infty,$$

since $a<\frac{1}{2}$.

Proof of Theorem 3.1. The proof of the asymptotic negligibility of the remainder terms B_n and C_n is given in Section 6. With an appropriate choice of the function J, Lemma 2.1 and Esseen's theorem imply $n^{1/2}(T_n-\mu_n)/\sigma_n$ and $n^{1/2}(U_n-\mu_n)/\sigma_n$ converge weakly to a standard normal distribution, provided $\liminf \sigma_n^2>0$. Finally, a lengthy algebra and Theorems 5.5 and 5.4 in Billingsley [3] show that $\sigma_n^2\to\sigma_0^2$ as $n\to\infty$.

Proof of Theorem 3.2. Let $L(x, y, d_1, d_2)$ be the joint distribution function of $(Y_{1i}, Y_{2i}, \delta_{1i}, \delta_{2i})$ and let $\hat{L}$ be the corresponding empirical distribution function. We can write

$$\begin{aligned}\hat{\sigma}_U^2-\sigma_U^2=4^{-1}\Big\{&\int \bar{J}^2(S)\,d(\hat{L}_1-L_1)+\int \bar{J}^2(S)\,d(\hat{L}_2-L_2)\\ &-2\int J(S(x),d_1)\,J(S(y),d_2)\,d(\hat{L}-L)(x,y,d_1,d_2)+D_n\Big\},\end{aligned} \tag{5.3}$$

where D_n is a remainder term. Similarly,

$$\begin{aligned}\hat{\sigma}_T^2-\sigma_T^2=4^{-1}\Big\{&\int d(\hat{L}_1-L_1)+\int d(\hat{L}_2-L_2)\\ &-2\int(-\ln(1-S(x))-d_1)(-\ln(1-S(y))-d_2)\\ &\times d(\hat{L}-L)(x,y,d_1,d_2)+E_n\Big\},\end{aligned} \tag{5.4}$$

where E_n is a remainder term. The asymptotic negligibility of the terms D_n and E_n is shown in Section 7. The leading terms are sums of iid mean zero rv's so that the conclusion follows from the law of large numbers.

The following lemma is needed to prove Corollaries 3.1 and 3.2. For $i=1,2$ let $\bar{H}_{ni}=1-H_{ni}$ and let

$$\Lambda_{ni}(t)=\int_0^t(\bar{F}_{ni})^{-1}\,dF_{ni}$$

be the cumulative hazard functions corresponding to cdf's F_{ni}.

LEMMA 5.1. *Let* (2.3) *and the assumptions of Theorem* 3.1 *be satisfied. Then*

$$\mu(F_n, G) = \int \tilde{J}(S_n)\, \bar{H}_{n1} \bar{H}_{n2} (H_{n1} + H_{n2})^{-1}\, d(\Lambda_{n1} - \Lambda_{n2}).$$

Here $\tilde{J}(u) = J(u, 1) - J(u, 0)$ *for* J *satisfying condition* A.2 *and* $\tilde{J}(u) \equiv 1$ *for the log-rank statistic.*

Proof. Equation (2.5) entails $J(u, 0) = 0$ for $u = 0$ and

$$\tilde{J}(u) = J(u, 1) - J(u, 0) = -(1-u)\, J'(u, 0).$$

Integration by parts and Lemma 41 yield for $i = 1, 2$,

$$\begin{aligned}
\mu_i &= \int J(S_n, 1)\, \bar{G}_{ni}\, dF_{ni} + \int J(S_n, 0)\, \bar{F}_{ni}\, dG_{ni} \\
&= \int \tilde{J}(S_n)\, \bar{G}_{ni}\, dF_{ni} + \int \bar{F}_{in} \bar{G}_{in} J'(S_n, 0)\, dS_n \\
&= \int \tilde{J}(S_n)\, \bar{G}_{ni}\, dF_{ni} + \int \bar{F}_{ni} \bar{G}_{ni} J'(S_n, 0)(1 - S_n)\, d\Lambda_n \\
&= \int \tilde{J}(S_n)\, \bar{G}_{ni}\, dF_{ni} - \int \tilde{J}(S_n)\, \bar{F}_{ni} \bar{G}_{ni}\, d\Lambda_n.
\end{aligned}$$

Using $dK_{ni} = \bar{G}_{ni}\, dF_{ni}$ and $\bar{H}_{ni} = \bar{F}_{ni} \bar{G}_{ni}$, we obtain

$$\mu_1 = \int \tilde{J}(S_n)\, \bar{H}_{1n} \bar{H}_{2n} (\bar{H}_{1n} + \bar{H}_{2n})^{-1}\, d(\Lambda_{1n} - \Lambda_{2n})$$

and

$$\mu_2 = \int \tilde{J}(S_n)\, \bar{H}_{1n} \bar{H}_{2n} (\bar{H}_{1n} + \bar{H}_{2n})^{-1}\, d(\Lambda_{2n} - \Lambda_{1n}).$$

The conclusion follows by noting that $\mu(F_n, G) = (\mu_2 - \mu_1)/2$.

Proof of Corollary 3.1. Consider a fixed alternative $F(s, t)$ such that $\lambda_1 \geqslant \lambda_2$. By Theorem 3.1, $n^{1/2}(U_n - \mu(F, G))$ and $n^{1/2}(T_n - \mu(F, G))$ converge weakly to mean zero normal distributions. Furthermore, by Lemma 5.1 $n^{1/2}\mu(F, G) \to \infty$. To complete the proof, it is enough to note that $\hat{\sigma}_U^2$ and $\hat{\sigma}_T^2$ converge in probability to a finite value. This can be established along the lines of the proof of Theorem 3.2.

Proof of Corollary 3.2. The proof follows directly from Theorem 3.1, Lemma 5.1, and some simple algebra.

6. Proof of Theorem 3.1: Remainder Terms

We now give the decomposition of the remainder terms B_n and C_n. Set $\Delta = [0, \max Y_{2i}]$ and $\Delta' = [0, \max Y_{1i}]$. The remainder term C_n in (3.2) is given by $C_n = C_{1n} + C_{2n}$, where

$$C_{1n} = 2^{-1} \int_{\Delta} n^{1/2}[J(\hat{S}(x), d) - J(S_n(x), d)]\, d\hat{L}_2(x, d) - A_{3n}$$

$$C_{2n} = -2^{-1} \int_{\Delta'} n^{1/2}[J(\hat{S}(x), d) - J(S_n(x), d)]\, d\hat{L}_1(x, d) - A_{4n}.$$

The remainder term B_n in (3.1) is given by $B_n = B_{1n} + B_{2n}$, where B_{kn} are defined as C_{kn} with $J(u, d) = -d - \ln(1-u)$ and $\hat{S}$ replaced by $1 - \exp(-\hat{\Lambda})$. The terms C_{1n} and C_{2n}, B_{1n} and B_{2n} are symmetric so in what follows we consider C_{1n} and B_{1n} only. For any $\tau \in (0, 1)$, let $A_\tau = [0, \gamma_\tau]$ where $\gamma_\tau = \inf\{s\colon H_2(s) \geqslant 1 - \tau\}$. Then $C_{1n} = \sum_{k=1}^{4} C_{1k}$ where

$$C_{11} = 2^{-1} \int_{\Delta \cap A_\tau} n^{1/2} W_n(x)(1 - S_n(x))\, J'(S_n(x), d)\, d(\hat{L}_2 - L_{n2})(x, d)$$

$$C_{12} = 2^{-1} \int_{\Delta^c \cup A_\tau^c} n^{1/2} W_n(x)(1 - S_n(x))\, J'(S_n(x), d)\, dL_{n2}(x, d)$$

$$\begin{aligned} C_{13} = 2^{-1} \int_{\Delta \cap A_\tau} & n^{1/2}[J(\hat{S}(x), d) - J(S_n(x), d) \\ & - (1 - S_n(x))\, W_n(x)\, J'(S_n(x), d)]\, d\hat{L}_2(x, d) \end{aligned}$$

$$C_{14} = 2^{-1} \int_{\Delta \cap A_\tau^c} n^{1/2}[J(\hat{S}(x), d) - J(S_n(x), d)]\, d\hat{L}_2(x, d).$$

Analogously, $B_{1n} = \sum_{k=1}^{4} B_{1k}$ where B_{1k} are defined as C_{1k} with $J(u, d) = -d - \ln(1-u)$ and $\hat{S}$ replaced by $1 - \exp(-\hat{\Lambda})$. The asymptotic negligibility of these terms will be proved by a sequence of lemmas showing that C_{11}, C_{13}, B_{11}, and B_{13} converge in probability to 0 for any fixed $\tau \in (0, 1)$ and $n \to \infty$, whereas the terms C_{12}, C_{14}, B_{12}, and B_{14} converge in probability to 0 as $\tau \downarrow 0$ and $n \to \infty$.

Lemma 6.1. *For fixed $\tau \in (0, 1)$, $C_{11} \to_P 0$ and $B_{11} \to_P 0$ as $n \to \infty$.*

Proof. Assuming that the function J satisfies assumption A.2 with $b \leqslant a + 1$, it is enough to consider the term C_{11} only. Let $\tau \in (0, 1)$ and $\varepsilon > 0$ be fixed. For any positive integer m define $\chi_m(x) = \gamma_\tau(k-1)/m$ for

$\gamma_\tau(k-1) < x \leqslant \gamma_\tau k/m$, $k=1, ..., m$. For arbitrary m, we have $|C_{11}| \leqslant \sum_{k=1}^{3} C_{11km}$, where

$$C_{111m} = \int_{\Delta \cap A_\tau} n^{1/2} \, |W_n(x) - W_n(\chi_m(x))| \, |\phi_n(x, d)| \, d(\hat{L}_2 + L_{n2})(x, d)$$

$$C_{1112m} = \int_{\Delta \cap A_\tau} n^{1/2} \, |W_n(\chi_m(x))| \, |\phi_n(x, d) - \phi_n(\chi_m(x), d)| \, d(\hat{L}_2 + L_{n2})(x, d)$$

$$C_{113m} = \left| \int_{\Delta \cap A_\tau} n^{1/2} W_n(\chi_m(x)) \, \phi_n(\chi_m(x), d) \, d(\hat{L}_2 - L_{n2})(x, d) \right|$$

and $\phi_n(x, d) = (1 - S_n(x)) \, J'(S_n(x), d)$.

There exists a constant $M_1 = M_1(\tau)$ such that for n large enough $\sup |S_n - S| < \tau/2$ and $\sup_{A_\tau} |\phi(\cdot, d)| < M_1$. Further, there exists a constant $M_2 = M_2(\tau, \varepsilon)$ such that for n sufficiently large the sets $\Omega_1 = \{\sup_{A_\tau} n^{1/2} \, |W_n| < M_2\}$ and $\Omega_2 = \{A_\tau \subset \Delta\}$ have probability at least $1 - \varepsilon$.

By Lemma 4.3, the process $n^{1/2} W_n$ converges weakly in $D(A_\tau)$ to a Gaussian process W. Therefore, by employing a Skorokhod construction, $\sup_{A_\tau} n^{1/2} \, |W_n - W_n \circ \chi_m| \to_P 0$ as $n, m \to \infty$ and there exists a sequence η_{mn}, $\eta_{mn} \to 0$ as $m, n \to \infty$, such that the set $\Omega_m = \{\sup_{A_\tau} |W_n - W_n \circ \chi_m| < \eta_{mn}\}$ has probability at least $1 - \varepsilon$ for all m and n sufficiently large. It follows that $I(\Omega_1 \cap \Omega_2 \cap \Omega_m) \, C_{111m} \leqslant M_1 \eta_{mn} \to 0$.

Further, for $d = 0, 1$ the function $J'(u, d)$ is uniformly continuous on $[0, 1 - \tau/2]$ so that for n sufficiently large $\xi_{mn} = \sup_{A_\tau} |\phi(x, d) - \phi(\chi_m(x), d)| \to 0$ as $m \to \infty$. It follows that $I(\Omega_1 \cap \Omega_2) \, C_{112m} \leqslant M_2 \xi_{mn} \to 0$ as $m, n \to \infty$.

Finally, for n sufficiently large, on the event $\Omega_1 \cap \Omega_2$ the integrand of C_{113m} is a step function assuming value a_{kmd} for $d = 0, 1$ and x belonging to $R_{km} = (\gamma_\tau(k-1)/m, \gamma_\tau k/m$, $k = 1, ..., m$. Therefore

$$\begin{aligned} I(\Omega_1 \cap \Omega_2) \, C_{113m} &= \left| \sum_{k=1}^{m} \sum_{d=0}^{1} a_{kmd} \int_{R_{km}} d(\hat{L}_2 - L_{n2}) \right| \\ &\leqslant 4mM_2(M_1 + \xi_{mn}) \sup |\hat{L}_2 - L_{n2}| \to_P 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Since $P(\Omega_1) > 1 - \varepsilon$ and $P(\Omega_2) > 1 - \varepsilon$ and ε was arbitrary, the conclusion follows.

LEMMA 6.2. *For fixed* $\tau \in (0, 1)$, $C_{13} \to_P 0$ *and* $B_{13} \to_P 0$ *as* $n \to \infty$.

Proof. By the mean value theorem, under condition A.2,

$$J(\hat{S}(x), d) - J(S_n(x), d) = (\hat{S}(x) - S_n(x)) \, J'(\Phi_d(x), d)$$

for $x \in \Delta$ and $d = 0, 1$. Here Φ_d is a random function assuming values between $\hat{S}(x)$ and $S_n(x)$. We can write $C_{13} = C_{131} + C_{132}$, where

$$C_{131} = 2^{-1} \int_{\Delta \cap A_\tau} n^{1/2} W_n(x)(1 - S_n(x)) \times [J'(\Phi_d(x), d) - J'(S_n(x), d)] \, d\hat{L}(x, d)$$

$$C_{132} = 2^{-1} \int_{\Delta \cap A_\tau} n^{1/2} [\hat{S}(x) - S_n(x) - W_n(x)(1 - S_n(x))] \times J'(\Phi_d(x), d) \, d\hat{L}(x, d).$$

Let $\tau \in (0, 1)$ and $\varepsilon > 0$ be fixed. For n sufficiently large $\sup |S_n - S| < \tau/3$. Further, there exist constants M_1 and M_2 such that for n sufficiently large, the sets $\Omega_1 = \{\sup\{|J'(\Phi_d(x), d)\colon \ x \in A_\tau, \ d = 0, 1\} < M_1\}$ and $\Omega_2 = \{\sup_{A_\tau} n^{1/2} |W_n| < M_2\}$ have probability at least $1 - \varepsilon$. For n sufficiently large, the sets $\Omega_3 = \{A_\tau \subset \Delta\}$ and $\Omega_4 = \{\sup_{A_\tau} |\hat{S} - S_n| < \tau/3\}$ have probability at least $1 - \varepsilon$.

We have

$$I(\Omega_1 \cap \Omega_2 \cap \Omega_4) \, |C_{131}| \leqslant M_2 [\sup_{A_\tau} |J'(\Phi_1(x), 1) - J'(S_n(x), 1)| + \sup_{A_\tau} |J'(\Phi_0(x), 0) - J'(S_n(x), 0)|].$$

For $d = 0, 1$, the function $J'(u, d)$ is uniformly continuous on $[0, 1 - \tau/3]$ so that $|\Phi_d - S_n| \leqslant |\hat{S} - S|$ and Lemma 4.2 imply that this bound converges in probability to 0 as $n \to \infty$. Further,

$$I\left(\bigcap_{k=1}^{4} \Omega_k\right) |C_{132}| \leqslant M_1 \sup_{A_\tau} n^{1/2} \, |\hat{S} - S_n - W_n(1 - S_n)|.$$

By Lemma 4.3 this bound converges in probability to 0. Since $P(\Omega_k) > 1 - \varepsilon$, $k = 1, 2, 3, 4$ and ε was arbitrary, it follows that $C_{13} \to_P 0$ as $n \to \infty$.

The proof of the asymptotic negligibility of B_{13} follows immediately from Lemmas 4.2 and 4.3.

LEMMA 6.3. *$C_{14} \to_P 0$ and $B_{14} \to_P 0$ as $\tau \downarrow 0$ and $n \to \infty$.*

Proof. Assuming that the function J satisfies condition A.2 with $b \leqslant a + 1$, it is enough to consider the term C_{14} only. We have $|C_{14}| \leqslant \sum_{k=1}^{4} C_{14k}$, where

$$C_{141} = \int_{\Delta^c \cup A_\tau^c} n^{1/2} \left(\int_0^x |\hat{H}_1^- - H_{n1}| \, r(H_n)^2 \, dH_n \right) r(H_n(x))^a \, dH_{n2}(x)$$

$$C_{142} = 2^{-1} \int_{\Delta^c \cup A_\tau^c} n^{1/2} \left(\int_0^x |\hat{H}_2^- - H_{n2}| \, r(H_n)^2 \, dH_n \right) r(H_n(x))^a \, dH_{n2}(x)$$

$$C_{143} = 2^{-1} \int_{\Delta^c \cup A_\tau^c} n^{1/2} \left| \int_0^x r(H_n) \, d(\hat{K}_1 - K_{n1}) \right| r(H_n(x))^a \, dH_{n2}(x)$$

$$C_{144} = 2^{-1} \int_{\Delta^c \cup A_\tau^c} n^{1/2} \left| \int_0^x r(H_n) \, d(\hat{K}_2 - K_{n2}) \right| r(H_n(x))^a \, dH_{n2}(x).$$

Let us consider the term C_{141}. Let $\varepsilon > 0$ and η, $0 < 2\eta < \frac{1}{2} - a$ be fixed. Corollary 1.1 in van Zuijlen [29], there exists $M_1 = M_1(\varepsilon)$ such that the set $\Omega_1 = \{\sup n^{1/2} \, |\hat{H}_1^- - H_{n1}| \, r H_{n1})^{1/2-\eta} < M_1\}$ has probability at least $1 - \varepsilon$ uniformly in n. Therefore,

$$\begin{aligned} I(\Omega_1) \, C_{141} &\leqslant 2^{-1} M_1 \int_{\Delta^c \cup A_\tau^c} \left(\int_0^x r(H_{n1})^{-1/2+\eta} \, r(H_n)^2 \, dH_n \right) \\ &\quad \times r(H_n(x))^a \, dH_{n2}(x) \\ &\leqslant M M_1 \int r(H_n)^{1-\eta} \, dH_n \int_{\Delta^c \cup A_\tau^c} r(H_{n2})^{a+1/2+\eta} \, dH_{n2} \qquad (6.1) \end{aligned}$$

for some constant M. The first integral in this bound does not depend on n and the underlying cdf's. To handle the second term, consider the integral

$$\int_{A_\tau^c} r(H_2)^{a+1/2+\eta} \, dH_2.$$

Applying the dominated convergence theorem, we can find $\tilde{\tau} = \tilde{\tau}(\varepsilon)$ such that for all $\tau \leqslant \tilde{\tau}$ this integral is less than $\varepsilon/2$. For this $\tilde{\tau}$ there exists $\tilde{n}$ such that the set $\Omega_2 = \{A_{\tilde{\tau}} \subset \Delta\}$ has probability at least $1 - \varepsilon$ and

$$\int_{A_\tau^c} [r(H_{n2})^{a+1/2+\eta} \, dH_{n2} - r(H_2)^{a+1/2+\eta} \, dH_2] < \varepsilon/2$$

for all $n \geqslant \tilde{n}$. It follows that the second integral in (6.1) is less than ε with probability at least $1 - \varepsilon$ for $\tau \leqslant \tilde{\tau}$ and $n \geqslant \tilde{n}$, i.e., $C_{141} \to_P 0$ as $\tau \downarrow 0$ and $n \to \infty$.

The proof of the asymptotic negligibility of the remaining terms is similar.

LEMMA 6.4. *For any* $0 \leqslant c < \frac{1}{2}$,

$$J_{1n} = \int_{\Delta \cap A_\tau^c} n^{1/2} |\hat{A} - A_n| \, r(H_{n2})^c \, d\hat{H}_2$$

$$J_{2n} = \int_{\Delta \cap A_\tau^c} n^{-1/2} \left(\int_0^x r(\hat{H}^-) \, r(\hat{H}^- - (2n)^{-1}) \, d\hat{K} \right) r(H_{n2}(x))^c \, d\hat{H}_2(x)$$

converge in probability to 0 *as* $\tau \downarrow 0$ *and* $n \to \infty$.

Proof. Let $\varepsilon > 0$ and η, $0 < 2\eta < \frac{1}{2} - c$ be fixed. We have

$$\int_{\Delta \cap A_\tau^c} r(H_{n2})^{c + 1/2 + 2\eta} \, d\hat{H}_2 \to_P 0. \tag{6.2}$$

as $\tau \downarrow 0$ and $n \to \infty$. This holds since

$$E \int_{\Delta \cap A_\tau^c} r(H_{n2})^{c + 1/2 + 2\eta} \, d\hat{H}_2 \leqslant \int_{A_\tau^c} r(H_{n2})^{c + 1/2 + 2\eta} \, dH_{n2}$$

and we can apply to this bound the arguments used in Lemma 6.3.

Let $A_\varepsilon = \{x\colon 1 - H_{n2}(x) > \varepsilon/n\}$ and $A'_\varepsilon = \{x\colon 1 - H_{r1}(x) > \varepsilon/n\}$. By Theorem 1.4 in van Zuijlen [29], the sets $\Omega_\varepsilon = \{\Delta \subset A_\varepsilon\}$ and $\Omega'_\varepsilon = \{\Delta' \subset A'_\varepsilon\}$ have probability at least $1 - \varepsilon$.

We have $I(\Omega_\varepsilon) \, J_{1n} \leqslant J_{11} + J_{12}$, where

$$J_{11} = n^{1/2} \int_{\Delta \cap A_\tau^c} \left(\int_0^x |\hat{H}^- - H_n| \, r(H_n) \, r(\hat{H}^-) \, d\hat{K} \right) r(H_{n2}(x))^c \, d\hat{H}_2(x)$$

$$J_{12} = n^{1/2} \int_{\Delta \cap A_\tau^c} \left| \int_0^x r(H_n) \, d(\hat{K}_1 - K_{n1}) \right| r(H_{n2}(x))^c \, d\hat{H}_2(x).$$

By Corollary 1.1 in van Zuijlen [29] there exists a constant $M_1 = M_1(\varepsilon)$ such that the sets $\Omega_1 = \{\sup n^{1/2} |\hat{H}_1^- - H_{n1}| \, r(H_{n1})^{1/2 - \eta} < M_1\}$ and $\Omega_2 = \{\sup n^{1/2} |\hat{H}_2^- - H_{n2}| \, r(H_{n2})^{1/2 - \eta} < M_1\}$ have probability at least $1 - \varepsilon/2$ uniformly in n. Set $M_2 = 2^{-1/2 - \eta} M_1$ and let $\Omega_3 = \{\sup n^{1/2} |\hat{H}^- - H_n| \, r(H_n)^{1/2 - \eta} < M_2\}$. Since $\Omega_1 \cap \Omega_2 \subset \Omega_3$, the set Ω_3 has probability at least $1 - \varepsilon$. We have

$$\begin{aligned}
I(\Omega_\varepsilon \cap \Omega_3) \, J_{11} &\leqslant M_2 \int_{\Delta \cap A_\tau^c} \left[\int_0^x r(H_n)^{1/2 + \eta} \, r(\hat{H}^-) \, d\hat{K} \right] r(H_{n2}(x))^c \, d\hat{H}_2 \\
&\leqslant 2^{-1} M_2 \int_{\Delta \cap A_\tau^c} \left[\int_0^x r(H_n)^{1/2 + \eta} \, r(\hat{H}^-) \, d\hat{H}_1 \right] r(H_{n2}(x))^c \, d\hat{H}_2 \\
&\quad + 2^{-1} M_2 \int_{\Delta \cap A_\tau^c} \left[\int_0^x r(H_n)^{1/2 + \eta} \, r(\hat{H}^-) \, d\hat{H}_2 \right] r(H_{n2}(x))^c \, d\hat{H}_2 \\
&= J_{111} + J_{112}.
\end{aligned}$$

By Theorem 1.1 in van Zuijlen [29], there exists a constant $M_3 = M_3(\varepsilon)$ such that the sets $\Omega_4 = \{\sup r(\hat{H}_1^-(u))\, r(H_{n1}(u))^{-1} < M_3,\ 0 \leqslant u \leqslant \max Y_{1i}\}$ and $\Omega_5 = \{\sup r(\hat{H}_2(u))\, r(H_{n2}(u))^{-1} < M_3,\ 0 \leqslant u \leqslant \max Y_{2i}\}$ have probability at least $1-\varepsilon$. Since $r(\hat{H}^-) \leqslant 2r(\hat{H}_i)$ and $r(H_n) \leqslant 2r(H_{ni})$, $i=1, 2$,

$$I\left(\bigcap_{k=3}^{5} \Omega_k \cap \Omega_\varepsilon\right) J_{11i} \leqslant M \int r(H_i^n)^{1-\eta}\, d\hat{H}_i \int_{\Delta \cap A_\tau^c} r(H_{n2})^{c+1/2+2\eta}\, d\hat{H}_2.$$

We have $\int r(H_{ni})^{1-\eta}\, d\hat{H}_i \rightarrow_P \int r(u)^{1-\eta}\, du$, so that (6.2) entails $J_{11} \rightarrow_P 0$ as $\tau \downarrow 0$ and $n \rightarrow \infty$. A similar argument, coupled with integration by parts, shows $J_{12} \rightarrow_P 0$ as $\tau \downarrow 0$ and $n \rightarrow \infty$. Further,

$$\begin{aligned}
&I\left(\bigcap_{k=3}^{5} \Omega_k \cap \Omega_\varepsilon\right) J_{2n} \\
&\quad \leqslant 2^{-1} \int_{\Delta \cap A_\tau^c} n^{-1/2} \left(\int_0^x r(\hat{H}^-)\, r(\hat{H}^- - (2n)^{-1})\, d\hat{H}_1\right) \\
&\quad \times r(H_{n2}(x))^c\, d\hat{H}_2(x) \\
&\quad + 2^{-1} \int_{\Delta \cap A_\tau^c} n^{-1/2} \left(\int_0^x r(\hat{H}^-)\, r(\hat{H}^- - (2n)^{-1})\, d\hat{H}_2\right) \\
&\quad \times r(H_{n2}(x))^c\, d\hat{H}_2(x) = J_{21} + J_{22}.
\end{aligned}$$

For some constant M, we have

$$\begin{aligned}
I(\Omega'_\varepsilon)\, J_{21} &\leqslant M n^{-1/2} \int_{A'_\varepsilon} r(H_{n1})^{3/2-2\eta}\, d\hat{H}_1 \int_{\Delta \cap A_\tau^c} r(H_{n2})^{c+1/2+2\eta}\, d\hat{H}_2 \\
&\leqslant M n^{-1/2} (n/\varepsilon)^{1/2-\eta} \int r(H_{n1})^{1/2-\eta}\, d\hat{H}_1 \int_{\Delta \cap A_\tau^c} r(H_{n2})^{c+1/2+2\eta}\, d\hat{H}_2
\end{aligned}$$

and

$$\begin{aligned}
J_{22} &\leqslant M n^{-1/2} \int_{A_\varepsilon} r(H_{n2})^{3/2-2\eta}\, d\hat{H}_2 \int_{\Delta \cap A_\tau^c} r(H_{n2})^{c+1/2+2\eta}\, d\hat{H}_2 \\
&\leqslant M n^{-1/2} (n/\varepsilon)^{1/2-\eta} \int r(H_{n2})^{1/2-\eta}\, d\hat{H}_2 \int_{\Delta \cap A_\tau^c} r(H_{n2})^{c+1/2+2\eta}\, d\hat{H}_2.
\end{aligned}$$

Since $\int r(H_{ni})^{1/2-\eta}\, d\hat{H}_i \rightarrow_P \int r(u)^{1/2-\eta}\, du$ (6.2) entails $J_{2n} \rightarrow_P 0$ as $\tau \downarrow 0$ and $n \rightarrow \infty$.

LEMMA 6.5. *$B_{14} \rightarrow_P 0$ and $C_{14} \rightarrow_P 0$ as $\tau \downarrow 0$ and $n \rightarrow \infty$.*

Proof. Setting $c=0$ in Lemma 6.4, we have $|B_{14}| \leqslant 2J_{1n}$ so that $B_{14} \to_P 0$ as $\tau \downarrow 0$ and $n \to \infty$.

Let Ω_ε and Ω_k, $k=1, ..., 5$ be defined as in Lemma 6.4. By the mean value theorem, condition A.2, Lemma 4.5, and in van Zuijlen [29],

$$I\left(\Omega_\varepsilon \cap \bigcap_{k=3}^{5} \Omega_k\right) C_{14} \leqslant n^{1/2} \int_{\Delta \cap A_\tau^c} |\hat{S}(x) - S_n(x)|\, |J'(\Phi_d(x), d)|\, d\hat{L}_2(x, d)$$

$$\leqslant n^{1/2} M \int_{\Delta \cap A_\tau^c} |\hat{S} - S_n|\, r(H_{n2})^b\, d\hat{L}_2$$

for some constant M. Applying inequalities $|x_1 - x_2| \leqslant |\ln x_1 - \ln x_2|$ for $0 < x_1$, $x_2 < 1$, and $0 < -\ln(1-(1+x)^{-1}) - (1+x)^{-1} < (x(x+1))^{-1}$ for $x > 0$, it can be verified that for $x \in \Delta$, on the set Ω_ε we have

$$|\hat{S}(x) - S_n(x)| \leqslant |\hat{\Lambda}(x) - \Lambda_n(x)| + n^{-1} \int_0^x r(\hat{H}^-)\, r(\hat{H}^- - (2n)^{-1})\, d\hat{K}.$$

Therefore $I(\bigcap_{k=3}^5 \Omega_k \cap \Omega_\varepsilon)\, C_{14} \leqslant M(J_{1n} + J_{2n})$, where J_{1n} and J_{2n} are defined as in Lemma 6.4 with $c=b$. It follows that $C_{14} \to_P 0$ as $\tau \downarrow 0$ and $n \to \infty$.

7. Proof of Theorem 3.2: Remainder Terms

We give the decomposition of the remainder terms D_n and E_n. As in Section 6, let $\Delta = [0, \max Y_{2i}]$ and $\Delta' = [0, \max Y_{1i}]$. The remainder term D_n in (5.3) is given by $D_n = \sum_{k=1}^4 D_{kn}$, where

$$D_{1n} = 4^{-1} \int_\Delta (\tilde{J}(\hat{S}) - \tilde{J}(S))\, d\hat{L}_2$$

$$D_{2n} = 4^{-1} \int_{\Delta'} (\tilde{J}(\hat{S}) - \tilde{J}(S))\, d\hat{L}_1$$

$$D_{3n} = -2^{-1} \int_{\Delta \times \Delta'} (J(\hat{S}(x), d_1) - J(S(x), d_1))\, J(\hat{S}(y), d_2)\, d\hat{L}(x, y, d_1, d_2)$$

$$D_{4n} = -2^{-1} \int_{\Delta \times \Delta'} (J(\hat{S}(x), d_1) - J(S(x), d_2))\, J(S(y), d_2)\, d\hat{L}(x, y, d_1, d_2).$$

The remainder term E_n in (5.4) is given by $E_n = \sum_{k=1}^4 E_{kn}$, where E_{kn} are defined as D_{kn} with $J(u, d) = -d - \ln(1-u)$ and $\hat{S}$ replaced by $1 - \exp(-\hat{\Lambda})$. The terms D_{1n} and D_{2n}, E_{1n} and E_{2n} are symmetric so in what follows, we consider D_{1n} and E_{1n} only.

For $\tau \in (0, 1)$, let $A_\tau = [0, \gamma_\tau]$, where $\gamma_\tau = \inf\{x: H_2(x) \geqslant 1 - \tau\}$. Then $D_{1n} = \sum_{k=1}^{3} D_{1k}$, where

$$D_{11} = 4^{-1} \int_{\Delta \cap A_\tau} (\tilde{J}(\hat{S}) - \tilde{J}(S))\, d\hat{L}_2$$

$$D_{12} = 4^{-1} \int_{\Delta \cap A_\tau^c} \tilde{J}(\hat{S})\, d\hat{L}_2$$

$$D_{13} = -4^{-1} \int_{\Delta \cap A_\tau^c} \tilde{J}(S)\, d\hat{L}_2.$$

LEMMA 7.1. *For fixed* $\tau \in (0, 1)$, $D_{11} \to_P 0$ *as* $n \to \infty$.

Proof. The function $\tilde{J}$ is uniformly continuous on A_τ. The conclusion follows from Lemma 4.2.

LEMMA 7.2. $D_{12} \to_P 0$ *and* $D_{13} \to_P 0$ *as* $\tau \downarrow 0$ *and* $n \to \infty$.

Proof. Let $\varepsilon > 0$ be fixed. Let $A_\varepsilon = \{x: 1 - H_2(x) > \varepsilon/n\}$. By Theorem 1.4 in van Zuijlen [29], the set $\Omega_\varepsilon = \{\Delta \subset A_\varepsilon\}$ has probability at least $1 - \varepsilon$. Further, by Theorem 1.1 in van Zuijlen [29], there exists a constant $M_1 = M_1(\varepsilon)$ such that the set $\Omega_1 = \{\sup_\Delta r(n\hat{H}_2/(n+1))\, r(H_2)^{-1} < M_1\}$ has probability at least $1 - \varepsilon$. Assumption A.2 and Lemma 4.1 entail

$$I(\Omega_\varepsilon \cap \Omega_1)\, D_{12} \leqslant M M_1 \int_{\Delta \cap A_\tau^c} r(H_2)^b\, d\hat{H}_2$$

$$D_{13} \leqslant M \int_{\Delta \cap A_\tau^c} r(H_2)^b\, d\hat{H}_2$$

for some constant M. As in Lemma 6.4, the bound converges in probability to 0 as $\tau \downarrow 0$ and $n \to \infty$.

LEMMA 7.3. $B_{1n} \to_P 0$ *as* $n \to \infty$.

Proof. We have $B_{1n} = n^{-1} \# \{i: \delta_{2i} = 1\} - P(\delta_2 = 1)$, so that the conclusion follows from the law of large numbers.

We proceed to consider terms D_{3n}, D_{4n}, E_{3n} and E_{4n}. Let $A'_\tau = [0, \gamma'_\tau]$, where $\gamma'_\tau = \inf\{x: H_1(x) \geqslant 1 - \tau\}$, and let $B_\tau = A'_\tau \times A_\tau$. Then $D_{3n} = \sum_{k=1}^{3} D_{3k}$, where

$$D_{31} = -2^{-1} \int_{\Delta' \times \Delta \cap B_\tau} [J(\hat{S}(x), d_1) - J(S(x), d_1)]\, J(\hat{S}(y), d_2)\, d\hat{L}(x, y, d_1, d_2)$$

$$D_{32} = -2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} J(\hat{S}(x), d_1)\, J(\hat{S}(y), d_2)\, d\hat{L}(x, y, d_1, d_2)$$

$$D_{33} = -2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} J(S(x), d_1)\, J(\hat{S}(y), d_2)\, d\hat{L}(x, y, d_1, d_2)$$

$$D_{41} = -2^{-1} \int_{\Delta' \times \Delta \cap B_\tau} [J(\hat{S}(x), d_1) - J(S(x), d_2)]\, J(S(y), d_2)\, d\hat{L}(x, y, d_1, d_2)$$

$$D_{42} = -2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} J(\hat{S}(x), d_1)\, J(S(y), d_2)\, d\hat{L}(x, y, d_1, d_2)$$

$$D_{43} = 2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} J(S(x), d_1)\, J(S(y), d_2)\, d\hat{L}(x, y, d_1, d_2).$$

The remainder terms E_{3n} and E_{4n} are given by $E_{3n} = \sum_{k=1}^{3} E_{3k}$ and $E_{4n} = \sum_{k=1}^{3} E_{4k}$, where E_{3k} and E_{4k} are defined as D_{3k} and D_{4k} with $J(u, d) = -d - \ln(1-u)$ and $\hat{S}$ replaced by $1 - \exp(-\hat{\Lambda})$.

LEMMA 7.4. *For fixed* $\tau \in (0, 1)$, D_{31}, D_{41}, E_{31}, *and* E_{41} *converge in probability to* 0 *as* $n \to \infty$.

Proof. The function $J(u, d)$ is uniformly continuous on A_τ and A'_τ. By Lemma 4.2, $D_{31} \to_P 0$ and $D_{41} \to_P 0$ as $n \to \infty$. The asymptotic negligibility of E_{31} and E_{41} follows directly from Lemma 4.2.

LEMMA 7.5. *The terms* D_{32}, D_{33}, D_{42}, *and* D_{43} *converge in probability to* 0 *as* $\tau \downarrow 0$ *and* $n \to \infty$.

Proof. We consider the term D_{32}. Let $\varepsilon > 0$ be fixed. Let A_ε and Ω_ε be defined as in Lemma 7.2. Further, let $A'_\varepsilon = \{x: 1 - H_1(x) > \varepsilon/n\}$. By Theorem 1.4 in van Zuijlen [29], the set $\Omega'_\varepsilon = \{\Delta' \subset A'_\varepsilon\}$ has probability at least $1 - \varepsilon$. Further, let M_1 and Ω_1 be defined as in Lemma 7.2. By Theorem 1.1 in van Zuijlen [29], there exists a constant $M_2 = M_2(\varepsilon)$ such that the set $\Omega_2 = \{\sup_{\Delta'} r(n\hat{H}_1/(n+1))\, r(H_1)^{-1} < M_2\}$ has probability at least $1 - \varepsilon$. Assumption A.2 and Lemma 4.1 entail

$$I(\Omega_\varepsilon \cap \Omega'_\varepsilon \cap \Omega_1 \cap \Omega_2)\, |D_{32}| \leqslant M \int_{\Delta \times \Delta' \cap B_\tau^c} r(H_1)^a\, r(H_2)^a\, d\hat{L}$$

for some constant M. By Hölder's inequality this bound is bounded from above

$$M \left[\int_{\Delta' \cap A_\tau'^c} r(H_1)^{2a}\, d\hat{H}_1 \int_{\Delta \cap A_\tau^c} r(H_2)^{2a}\, d\hat{H}_2 \right]^{1/2}$$
$$+ M \left[\int_{\Delta' \cap A_\tau'^c} r(H_1)^{2a}\, d\hat{H}_1 \int_{\Delta \cap A_\tau} r(H_2)^{2a}\, d\hat{H}_2 \right]^{1/2}$$
$$+ M \left[\int_{\Delta' \cap A'_\tau} r(H_1)^{2a}\, d\hat{H}_1 \int_{\Delta \cap A_\tau^c} r(H_2)^{2a}\, d\hat{H}_2 \right]^{1/2}.$$

As in Lemma 6.4, $\int_{A_\tau^c} r(H_2)^{2a}\, d\hat{H}_2 \to_P 0$ and $\int_{A_\tau^c} r(H_1)^{2a}\, d\hat{H}_1 \to_P 0$ as $\tau \downarrow 0$ and $n \to \infty$, which completes the proof of the asymptotic negligibility of D_{32}. The remaining terms can be treated in a similar way.

LEMMA 7.6. *The terms E_{32}, E_{33}, E_{42}, and E_{43} converge in probability to 0 as $\tau \downarrow 0$ and $n \to \infty$.*

Proof. We consider the term E_{32}. We have $|E_{32}| \leqslant \sum_{k=1}^{4} E_{32k}$, where

$$E_{321} = 2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} \hat{\Lambda}(x)\, \hat{\Lambda}(y)\, d\hat{L}(x, y, d_1, d_2)$$

$$E_{322} = 2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} \hat{\Lambda}(x)\, d\hat{L}(x, y, d_1, d_2)$$

$$E_{323} = 2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} \hat{\Lambda}(y)\, d\hat{L}(x, y, d_1, d_2)$$

$$E_{324} = 2^{-1} \int_{\Delta' \times \Delta \cap B_\tau^c} d\hat{L}(x, y, d_1, d_2).$$

Let $\varepsilon > 0$ and $a < \frac{1}{2}$ be fixed. Further, let Ω_ε, Ω'_ε, Ω_1, and Ω_2 be defined as in Lemma 7.5. Then

$$I(\Omega_\varepsilon \cap \Omega'_\varepsilon \cap \Omega_1 \cap \Omega_2)\, E_{321}$$
$$\leqslant M \left[\int r(\hat{H}^-)^{1-a}\, d\hat{H} \right]^2 \times \int_{\Delta' \times \Delta \cap B_\tau^c} r(H_1)^a\, r(H_2)^a\, d\hat{L}$$

for some constant M. Since $\int r(\hat{H}^-)^{1-a}\, d\hat{H} \to_P \int r(u)^{1-a}\, du < \infty$, the same argument as in Lemma 7.5 entails $E_{321} \to_P 0$ as $\tau \downarrow 0$ and $n \to \infty$. The asymptotic negligibility of terms E_{32k}, $k = 2, 3, 4$ follows from a similar argument combined with the fact that the $\hat{L}_1$ measure of $\Delta' \cap A_\tau^{1c}$ and $\hat{L}_2$ measure of $\Delta \cap A_\tau^c$ converges in probability to 0 as $\tau \downarrow 0$ and $n \to \infty$. The proof of the asymptotic negligibility of the remaining terms follows in a similar fashion.

ACKNOWLEDGMENT

I thank Professor Kjell Doksum for helpful discussions.

REFERENCES

[1] AALEN, O. O. (1978). Nonparametric inference for a family of counting processes. *Ann. Statist.* **6** 701–726.

[2] BELL, C. B., AND HALLER, S. H. (1969). Bivariate symmetry tests: Parametric and nonparametric. *Ann. Math. Statist.* **40** 259–269.

[3] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[4] BRESLOW, N., AND CROWLEY, J. J. (1974). A large sample study of the life table and product limit estimates under random censorship. *Ann. Statist.* **2** 437–453.

[5] CAMPBELL, G. (1981). Nonparametric bivariate estimation with randomly censored data. *Biometrika* **68** 417–422.

[6] CAMPBELL, G. (1982). Asymptotic properties of several nonparametric multivariate distribution function estimators under random censoring. In *Survival Analysis* (J. J. Crowley and R. A. Johnson, Eds.), pp. 243–256. IMS Lecture Notes, Monograph Series 2. Inst. Math. Statist., Hayward, CA.

[7] CHERNOFF, H., AND SAVAGE, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* **29** 972–994.

[8] CLAYTON, D. G. (1978). A model of association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika* **65** 141–151.

[9] CLAYTON, D. G., AND CUZICK, J. (1985). Multivariate generalization of the proportional hazards model. *J. Roy. Statist. Soc. Ser. A* **148** 82–117.

[10] CROWLEY, J. J. (1973). *Nonparametric Analysis of Censored Survival Data, with Distribution Theory for the k-Sample Generalized Savage Statistic.* Ph.D. thesis, University of Washington.

[11] CUZICK, J. (1985). Asymptotic properties of censored linear rank tests. *Ann. Statist.* **13** 133–141.

[12] DABROWSKA, D. M. (1986). Rank tests for independence for bivariate censored data. *Ann. Statist.* **14** 250–264.

[13] DOKSUM, K. A. (1980). Rank tests for the matched pair problem with life distributions. *Scand. J. Statist.* **7** 67–72.

[14] GILL, R. D. (1980). *Censoring and Stochastic Integrals.* Mathematical Centre Tracts Vol. 124. Math. Centrum, Amsterdam.

[15] HÁJEK, J. A., AND ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academic Press, New York.

[16] KALBFLEISCH, J. D., AND PRENTICE, R. S. (1980). *The Statistical Analysis of Failure Time Data.* Wiley, New York.

[17] KAPLAN, E. L., AND MEIER, P. (1958). Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.* **53** 457–481.

[18] LANGBERG, N. A., AND SHAKED, M. (1982). On the identifiability of multivariate life distribution functions. *Ann. Probab.* **10** 773–779.

[19] LIPTSER, R. S., AND SHIRYAEV, A. N. (1978). *Statistics of Random Processes II: Applications.* Springer-Verlag, New York.

[20] NELSON, W. (1972). Theory and applications of hazard plotting for censored failure data. *Technometrics* **14** 945–966.

[21] OAKES, D. (1982). A model for association in bivariate survival data. *J. Roy. Statist. Soc. Ser. B* **44** 414–422.

[22] PRENTICE, R. S. (1978). Linear rank tests with censored data. *Biometrika* **65** 167–179.

[23] PYKE, R., AND SHORACK, G. R. (1968). Weak convergence of a two-sample empirical process and a new approach to Chernoff–Savage theorems. *Ann. Math. Statist.* **39** 755–771.

[24] SCHAAFSMA, W. (1976). *Bivariate Symmetry and Asymmetry*. Tech. Report, University of Groningen, The Netherlands.

[25] SHORACK, G. R., AND WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.

[26] SNIJDERS, T. (1976). *Tests for the Problem of Bivariate Symmetry*. Tech. Report, University of Groningen, The Netherlands.

[27] SNIJDERS, T. (1981). Rank tests for bivariate symmetry. *Ann. Statist.* **9** 1087–1095.

[28] TSAI, W. Y., LEURGANS, S., AND CROWLEY, J. J. (1986). Nonparametric estimation of a bivariate survival function in the presence of censoring. *Ann. Statist.* **14** 1351–1365.

[29] VAN ZUIJLEN, M. C. A. (1978). Properties of the empirical distribution functions for independent nonidentically distributed random variables. *Ann. Probab.* **6** 250–266.

[30] WEI, L. J., AND PEE, D. (1985). Distribution-free methods for estimating location difference with censored paired data. *J. Amer. Statist. Assoc.* **80** 405–410.

[31] YANAGIMOTO, T., AND SIBUYA, M. (1976). Test of symmetry of a bivariate distribution. *Sankhyā Ser. A* **38** 105–115.